

3. Komplexe Zahlen

<https://www.youtube.com/watch?v=-vd4RLw379Q>

$$x^2 + 4 = 0$$

$$x^2 = -4 \quad | : \sqrt{}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \pm \sqrt{-4} \Rightarrow \pm \sqrt{(-1) \cdot 4} = \pm \sqrt{4} \cdot \underbrace{\sqrt{-1}}_i = \pm 2i$$

Definiere $\sqrt{-1} = i \Leftrightarrow i^2 = -1$

$$\bullet \quad 3 + \sqrt{8} = 3 + \sqrt{4 \cdot 2} = 3 + 2\sqrt{2}$$

$$\bullet \quad 3 + \sqrt{-2} = 3 + \sqrt{2}i$$

Komplexe Zahlen: $z \in \mathbb{C}$

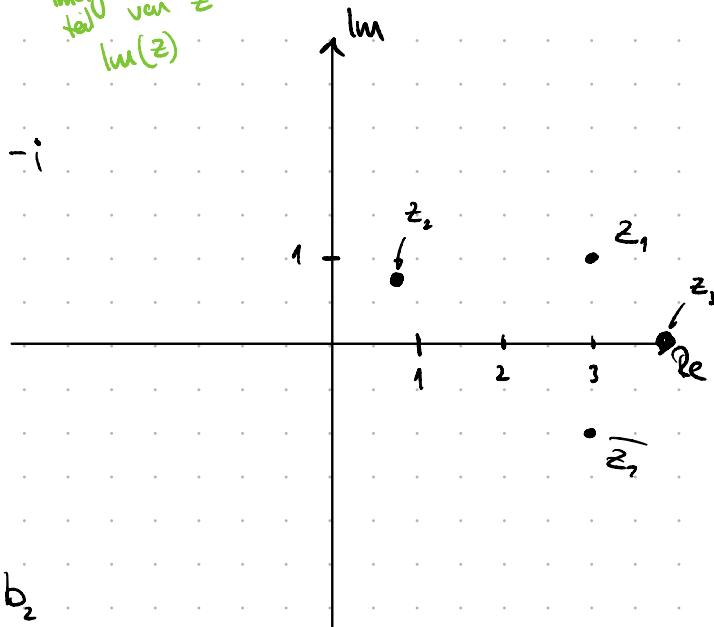
$$z = \underbrace{a}_{\text{Realteil von } z} + i \underbrace{b}_{\text{Imaginärteil von } z} \quad a, b \in \mathbb{R}$$

$\text{Re}(z)$ $\text{Im}(z)$

$$z_1 = 3 + i \quad \Leftrightarrow \quad \bar{z}_1 = 3 - i$$

$$z_2 = \frac{1}{\sqrt{2}}(1 + i)$$

$$z_3 = 4$$



Rechenregeln

$$z_1 = a_1 + ib_1 \quad z_2 = a_2 + ib_2$$

$$\bullet \quad z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$\bullet \quad z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + i a_1 b_2 + i a_2 b_1 - b_1 b_2$$

$$\text{Re}(z_1 z_2) = a_1 a_2 - b_1 b_2$$

$$\text{Im}(z_1 z_2) = a_1 b_2 + a_2 b_1$$

$$\bullet \quad \frac{1}{z_1} = \frac{1}{a_1 + ib_1} \Rightarrow \frac{1}{z_1} \cdot \frac{\bar{z}_1}{\bar{z}_1} = \frac{\bar{z}_1}{\underbrace{z_1 \bar{z}_1}_{\in \mathbb{R}}}$$

- Komplex Konjugierte einer Zahl $z \in \mathbb{C}$

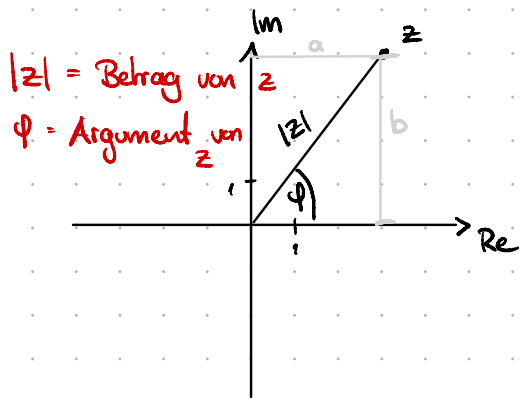
$$z = a + ib \iff \bar{z} = a - ib$$

$$\bar{z} \cdot z = (a + ib)(a - ib) = a^2 + \cancel{iab} - \cancel{iab} - \underbrace{(ib)^2}_{-b^2} = a^2 + b^2$$

$$\frac{1+3i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+3i)(1+i)}{(1-i)(1+i)} = \frac{(1+3i)(1+i)}{2} = \frac{1+3i+i-3}{2} = \frac{-2+4i}{2} = -1+2i$$

Polarkoordinaten

$$z = a + ib$$



$|z|$ = Abstand von z zum Ursprung

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z \bar{z}} \iff z \bar{z} = |z|^2$$

$$\varphi = \arctan\left(\frac{\text{Im}}{\text{Re}}\right) = \arctan\left(\frac{b}{a}\right)$$

$$a = |z| \cdot \cos \varphi$$

$$b = |z| \cdot \sin \varphi$$

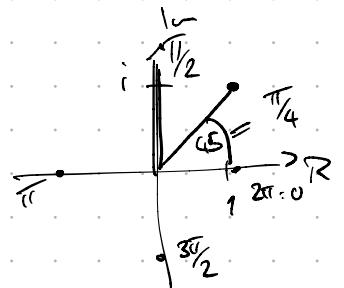
$$\Rightarrow z = |z| \cdot (\cos \varphi + i \sin \varphi)$$

$$z_1 = |z_1| (\cos \varphi_1 + i \sin \varphi_1) \quad z_2 = |z_2| (\cos \varphi_2 + i \sin \varphi_2)$$

$$z_1 \cdot z_2 = |z_1| |z_2| (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$$

„Beträge werden multipliziert, Winkel addiert“

$$z_1^n = |z_1|^n \cos(n \cdot \varphi_1) + i \sin(n \cdot \varphi_1)$$



Komplexes Wurzelziehen

$$w^n = z \quad \text{mit} \quad z = a + ib \Rightarrow |z| \cdot (\cos \varphi + i \sin \varphi)$$

$$w_k = \sqrt[n]{|z|} \left(\cos\left(\frac{\varphi}{n} + \frac{2\pi}{n} \cdot k\right) + i \sin\left(\frac{\varphi}{n} + \frac{2\pi}{n} \cdot k\right) \right) \quad k = 0, \dots, n-1$$

Bsp $w^4 = 1 + i \Rightarrow z = \sqrt{2} \cdot (\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4}))$

$$w_k = \sqrt[4]{\sqrt{2}} \left(\cos\left(\frac{\pi}{16} + \frac{\pi}{2} \cdot k\right) + i \sin\left(\frac{\pi}{16} + \frac{\pi}{2} \cdot k\right) \right)$$

$$\omega_0 = \sqrt[4]{2} \left(\cos\left(\frac{\pi}{16} + \frac{\pi}{2} \cdot 0\right) + i \sin\left(\frac{\pi}{16} + \frac{\pi}{2} \cdot 0\right) \right)$$

$$= \sqrt[4]{2} \left(\cos\left(\frac{\pi}{16}\right) + i \sin\left(\frac{\pi}{16}\right) \right)$$

$$\omega_1 = \sqrt[4]{2} \left(\cos\left(\frac{\pi}{16} + \frac{\pi}{2} \cdot 1\right) + i \sin\left(\frac{\pi}{16} + \frac{\pi}{2}\right) \right)$$

$$= \sqrt[4]{2} \left(\cos\left(\frac{9\pi}{16}\right) + i \sin\left(\frac{9\pi}{16}\right) \right)$$

$$\omega_2 =$$

$$\omega_3 =$$

Quadrant	1	1	1	1	1/2	2	2	2	2/3	3	3	3	3/4
Winkel	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°
Winkel	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π	$-\frac{5}{6}\pi$	$-\frac{3}{4}\pi$	$-\frac{2}{3}\pi$	$-\frac{1}{2}\pi$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

Quadrant	4	4	4	4
Winkel	300°	315°	330°	360°
Winkel	$-\frac{1}{3}\pi$	$-\frac{1}{4}\pi$	$-\frac{1}{6}\pi$	0
sin	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

$$\arctan\left(\frac{\text{Im}}{\text{Re}}\right) = \arctan\left(\frac{b}{a}\right) = \varphi$$

$$\tan(\varphi) = \frac{b}{a}$$

$$z_1 = 1 - i\sqrt{3}$$

$$z_4 = 2 - i$$

$$z_2 = 1 - i$$

$$z_5 = 3 - i$$

$$z_3 = -2 - 2i\sqrt{3}$$

$$\varphi_2 \neq \frac{b}{a} = \frac{-1}{1} = -1 \quad z_2 \text{ in 4. Quadrant}$$

$$\Rightarrow \varphi_2 = \frac{-\pi}{4} = \frac{7\pi}{8}$$

$$\varphi_3: \frac{-2\sqrt{3}}{2} = \sqrt{3} \Rightarrow \varphi = \frac{-2\pi}{3} = \frac{4\pi}{3}$$

$$\varphi_4: \frac{b}{a} = \frac{-1}{2} = -\frac{1}{2}$$

